

ZEROS OF ENTIRE FUNCTIONS

BY
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Introduction. This paper is the outgrowth of the author's Ph.D. thesis written under the supervision of Professor A. Peyerimhoff, whose paper on zeros of power series [4] has been the starting point.

§1 contains some lemmas due to A. Peyerimhoff, which are needed in §2.

In all theorems (with the exception of Theorem 4 only) of the remaining part of this paper, real entire functions $f(z)$ of finite order with infinitely many real and finitely many complex⁽¹⁾ zeros are taken as starting point.

Then in §2 functions of the form $g(z) = A(z)f(z) + B(z)f'(z)$ are considered where $A(z)$ and $B(z)$ are real polynomials. In Theorem 1 upper bounds for the number of zeros of $g(z)$ in the complex plane with the exception of certain real intervals are obtained. In certain instances the *exact* number of zeros is obtained. At the end of §2 examples are given which show that the results of Theorem 1 are best possible. Questions of this kind have been considered by Laguerre and Borel especially [1], [2] for the particular case $g(z) = f'(z)$. One of the results of [4] deals with functions $g(z) = \alpha f(z) + zf'(z)$ where α is real and $f(z)$ of order < 1 . The method of proof of this result has been generalized in the present paper.

In §3 a couple of theorems (2, 3, 5, 6) which are derived from Theorem 1 are partly slight generalizations of known theorems, especially of Laguerre [2], [3], [5], [6]. These theorems deal with questions of the following kind. If one has some information on the zeros of the entire function $f(z) = \sum a_n z^n$, one wants to gain information on the zeros of $F(z) = \sum a_n G(n) z^n$, where $G(z)$ is an entire function of a certain type.

For Theorem 4, which is known already, a short proof is given here which is independent of Theorem 1. From this theorem follows immediately the well-known fact that the Besselfunctions of real order > -1 have real zeros only.

From Theorem 6 Hurwitz's Theorem on the complex zeros of the Besselfunctions of real order < -1 follows as a special case.

In Theorem 9 functions $g(z) = \alpha f(z) + z^l f'(z)$ are considered, where α is real, l odd, and $f(z)$ is a real canonical product of finite genus. Then the *exact* number of complex zeros of $g(z)$ is obtained. Especially we find that for $\alpha > 0$ and l odd the functions $\alpha \sin z + z^l \cos z$ and $\alpha \cos z - z^l \sin z$ have *exactly* $l-1$ complex zeros.

1. In the proof of Lemma 1 we will need some elementary properties of difference quotients. In the following we suppose that $f(z)$ is a holomorphic function.

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⁽¹⁾ Here "complex" means "nonreal".

Difference quotients which are undefined shall be defined by the corresponding differential quotients.

Let

$$\Delta_{z_1}f(z) = \frac{f(z) - f(z_1)}{z - z_1}$$

and recursively we define

$$\Delta_{z_l \dots z_1}f(z) = \Delta_{z_l}(\Delta_{z_{l-1} \dots z_1}f(z)).$$

This operation on functions $f(z)$ is homogeneous and linear, and for products $f(z) = g(z) \cdot h(z)$ we have $\Delta_{z_1}(g(z)h(z)) = g(z_1)\Delta_{z_1}h(z) + h(z)\Delta_{z_1}g(z)$ and, as can be proved by induction on l :

$$(1) \quad \begin{aligned} \Delta_{z_l \dots z_1}(g(z)h(z)) &= g(z_1)\Delta_{z_l \dots z_1}h(z) + h(z)\Delta_{z_l \dots z_1}g(z) \\ &+ \sum_{n=1}^{l-1} (\Delta_{z_n \dots z_1}g(z))|_{z=z_{n+1}} \Delta_{z_l \dots z_{n+1}}h(z). \end{aligned}$$

Let C_1 denote the exterior of the interval $[1, +\infty)$ with respect to the finite complex plane and C_2 the exterior of the intervals $(-\infty, -1]$ and $[1, +\infty)$ with respect to the finite complex plane.

Then we can prove the following lemma due to A. Peyerimhoff [4] for which a different proof is given here:

LEMMA 1. *Let $f(z) = P_{k-1}(z) + z^k \int_{-1}^1 (1/(1-zt)) dg(t)$, where $P_{k-1}(z)$ is a real polynomial of degree $k-1$ at most ($P_{k-1}(z) \equiv 0$ if $k=0$) and $g(t)$ is real and weakly monotone on $-1 \leq t \leq 1$ with $g(1) \neq g(-1)$.*

Then $f(z)$ is holomorphic and has at most k zeros in C_2 . Similarly if $f(z) = P_{k-1}(z) + z^k \int_0^1 (1/(1-zt)) dg(t)$, where $g(t)$ is real and weakly monotone on $0 \leq t \leq 1$, then $f(z)$ is holomorphic and has at most k zeros in C_1 .

Proof. We give a proof for the first part, the proof for the second part being similar.

If $k=0$, then $f(z) = \int_{-1}^1 (1/(1-zt)) dg(t)$.

If z is real, $z=x$, $f(x) = \int_{-1}^1 (1/(1-xt)) dg(t) \neq 0$ for $-1 < x < 1$ since $g(1) \neq g(-1)$.

If $z = x + iy$ with $y \neq 0$, consider

$$zf(z) = \int_{-1}^1 \frac{z(1-\bar{z}t)}{|1-zt|^2} dg(t) = \int_{-1}^1 \frac{x-|z|^2t}{|1-zt|^2} dg(t) + iy \int_{-1}^1 \frac{1}{|1-zt|^2} dg(t).$$

Then

$$\operatorname{Im}(zf(z)) = y \int_{-1}^1 \frac{1}{|1-zt|^2} dg(t) \neq 0,$$

since $y \neq 0$. Therefore $zf(z) \neq 0$ for $z \neq 0$ and $f(0) = g(1) - g(-1) \neq 0$. Therefore, if $k=0$, $f(z)$ has no zeros in C_2 .

Assume now $k \geq 1$. Then $f(z) = P_{k-1}(z) + z^k \int_{-1}^1 (1/(1-zt)) dg(t)$ and we prove by induction on k that for $z, z_1, z_2, \dots, z_k \in C_2$ we have

$$(2) \quad \Delta_{z_k \dots z_1} f(z) = \int_{-1}^1 \frac{1}{(1-zt)(1-z_1t) \dots (1-z_kt)} dg(t).$$

This is trivial for $k=1$ and $\Delta_{z_k \dots z_1}(P_{k-1}(z))=0$ is trivial for any k .

Assume now that (2) holds for $k-1$ instead of k . Then

$$\Delta_{z_{k-1} \dots z_1} \left(\frac{z^{k-1}}{1-zt} \right) = \frac{1}{(1-zt)(1-z_1t) \dots (1-z_{k-1}t)}.$$

Now apply (1) with $h(z)=z$ and $g(z)=z^{k-1}/(1-zt)$. Then

$$\begin{aligned} \Delta_{z_k \dots z_1} \left(\frac{z^{k-1}}{1-zt} \right) &= \left(\Delta_{z_{k-1} \dots z_1} \left(\frac{z^{k-1}}{1-zt} \right) \right) \Big|_{z=z_k} \Delta_{z_k}(z) + z \Delta_{z_k \dots z_1} \left(\frac{z^{k-1}}{1-zt} \right) \\ &= \frac{1}{(1-z_kt)(1-z_1t) \dots (1-z_{k-1}t)} + \frac{zt}{(1-zt)(1-z_1t) \dots (1-z_kt)} \\ &= \frac{1}{(1-zt)(1-z_1t) \dots (1-z_kt)} \quad \text{which proves (2).} \end{aligned}$$

Now we assume that $f(z) = P_{k-1}(z) + z^k \int_{-1}^1 (1/(1-zt)) dg(t)$ has at least k zeros $z_1, \dots, z_k$ in C_2 . Then

$$(3) \quad \Delta_{z_k \dots z_1} f(z) = \frac{f(z)}{(z-z_1) \dots (z-z_k)} = \int_{-1}^1 \frac{1}{(1-zt)(1-z_1t) \dots (1-z_kt)} dg(t).$$

If k is even we may assume that $z_1, \dots, z_k$ consists of pairs of conjugate complex zeros and of real zeros of $f(z)$ in C_2 . Then (3) gives:

$$\frac{f(z)}{(z-z_1) \dots (z-z_k)} = \int_{-1}^1 \frac{1}{1-zt} d\gamma(t), \quad \text{where } \gamma(t) = \int_{-1}^t \frac{dg(\tau)}{(1-z_1\tau) \dots (1-z_k\tau)}$$

is a real monotone function for $-1 \leq t \leq 1$. According to what has been proved already $\int_{-1}^1 (1/(1-zt)) d\gamma(t)$ has no zeros in C_2 and so $f(z)$ does not have more zeros than $z_1, \dots, z_k$.

If k is odd and $f(z)$ has at least k zeros in C_2 , then $f(z)$ has at least 1 real zero in C_2 . Otherwise $f(z)$ would have at least $k+1$ complex zeros $z_1, \dots, z_k, z_{k+1}$ in C_2 which we can assume to consist of pairs of conjugate complex zeros. Especially we assume that $z_{k+1} = \bar{z}_k$. Then (3) gives.

$$(4) \quad \frac{f(z)}{(z-z_1) \dots (z-z_k)} = \int_{-1}^1 \frac{1}{(1-zt)(1-z_kt)} d\gamma(t)$$

where

$$\gamma(t) = \int_{-1}^t \frac{dg(\tau)}{(1-z_1\tau) \dots (1-z_{k-1}\tau)}$$

is a real monotone function for $-1 \leq t \leq 1$.

In (4) the left side is holomorphic in C_2 and vanishes at $z = z_{k+1} = \bar{z}_k$ whereas the right side is $\int_{-1}^1 (1/|1 - z_k t|^2) d\gamma(t) \neq 0$. Therefore, if $f(z)$ has at least k zeros in C_2 , where k is odd, $f(z)$ has at least one real zero in C_2 . Now we apply the same argument as in the case where k was even to the k zeros $z_1, \dots, z_k$ in C_2 , where we now can assume that $z_1, \dots, z_k$ (k odd) consists of pairs of conjugate complex and of real zeros (at least one) of $f(z)$ in C_2 . This again shows that $f(z)$ cannot have more than k zeros in C_2 .

LEMMA 2. *If $P_l(z)$ is a real polynomial of degree l and $g(t)$ is real and weakly monotone for $-1 \leq t \leq 1$, then for $z \in C_2$*

$$P_l(z) \int_{-1}^1 \frac{1}{1-zt} dg(t) = P_{l-1}(z) + z^l \int_{-1}^1 \frac{t^l P_l(1/t)}{1-zt} dg(t)$$

where $P_{l-1}(z)$ is a real polynomial of degree $l-1$ at most.

A similar formula holds if the integrals from -1 to 1 are replaced by integrals from 0 to 1 , and if C_2 is replaced by C_1 .

Proof⁽²⁾. $h(z) = P_l(z) - z^l t^l P_l(1/t)$ is a polynomial in z of degree l at most which vanishes at $z = 1/t$. Therefore $h(z)/(1-zt)$ is a real polynomial in z of degree $l-1$ at most.

The same is true of $\int_{-1}^1 (h(z)/(1-zt)) dg(t)$.

LEMMA 3. *If $l < k$ and $g(t)$ is real and weakly monotone for $-1 \leq t \leq 1$ then*

$$z^l \int_{-1}^1 \frac{1}{1-zt} dg(t) = P_{k-1}(z) + z^k \int_{-1}^1 \frac{t^{k-l}}{1-zt} dg(t) \quad \text{for } z \in C_2.$$

A similar formula holds for $z \in C_1$ if the integrals from -1 to 1 are replaced by integrals from 0 to 1 .

Proof⁽²⁾. $h(z) = z^l - z^k t^{k-l}$ is a real polynomial in z of degree k at most which vanishes at $z = 1/t$.

Therefore $h(z)/(1-zt)$ is a real polynomial in z of degree $k-1$ at most. The same is true of $\int_{-1}^1 (h(z)/(1-zt)) dg(t)$.

2. We assume that $f(z)$ is an entire function of finite order ρ which is real for real z and which has finitely many complex zeros and infinitely many real zeros.

If r_n ($n=1, 2, \dots$) are the absolute values in increasing order of the zeros $z_n \neq 0$ of $f(z)$ then there exists a smallest nonnegative integer p (the genus of the sequence r_n) such that $\sum_{n=1}^{\infty} 1/r_n^p = \infty$ and $\sum_{n=1}^{\infty} 1/r_n^{p+1} < \infty$. Then always $p \leq \rho$ and $p = [\rho]$ if ρ is not an integer [6]. The convergence exponent σ of r_n is the greatest lower bound of positive numbers α such that $\sum_{n=1}^{\infty} 1/r_n^\alpha < \infty$. Then always $p \leq \sigma \leq p+1$.

(2) I am indebted to the referee for simplifying the proofs of Lemmas 2 and 3.

From Hadamard's factorization theorem [6] follows that $f(z)$ can be written as

$$(1) \quad f(z) = z^k e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp \left[\frac{z}{z_n} + \frac{(z/z_n)^2}{2} + \cdots + \frac{(z/z_n)^p}{p} \right]$$

where $Q(z)$ is a real polynomial of degree q with $q \leq [\rho]$ and

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp \left[\frac{z}{z_n} + \cdots + \frac{(z/z_n)^p}{p} \right]$$

is the canonical product of genus p associated with the zeros z_n which, as an entire function is of order σ [6]. Then always $\rho = \max(q, \sigma)$ [6].

Now we consider the entire function (of order $\leq \rho$).

$g(z) = A(z)f(z) + B(z)f'(z)$ where $A(z)$ and $B(z)$ are real polynomials of degrees a and b respectively, where we agree that always $b \geq 0$ and $A(z) \equiv 0$ if $a = -1$.

With this notation we have the following

THEOREM 1.

Case I. Let $f(z)$ have infinitely many positive, finitely many negative and finitely many complex zeros ($f(z)$ may have a zero at $z=0$).

Let a_1 denote a real zero of $f(z)$ with multiplicity α_1 such that all real zeros of $B(z)$ are less than a_1 . All real zeros of $f(z)$ which are bigger than a_1 are denoted by a_n in increasing order with multiplicities α_n ($n=2, 3, \dots$). If $B(z)$ does not have any real zeros, a_1 can be any real zero of $f(z)$.

Then at each point a_n ($n=1, 2, \dots$), $g(z)$ has a zero of multiplicity $\alpha_n - 1$ exactly and between each pair a_n, a_{n+1} ($n=1, 2, \dots$) $g(z)$ has an odd number of zeros.

Let k be the number of zeros of $f(z)$ which are different from a_n ($n=1, 2, \dots$). Then the number of zeros of $g(z)$ besides the trivial zeros at a_n and besides one zero between each pair a_n, a_{n+1} ($n=1, 2, \dots$) is

1. exactly $b + p + k$ if $p \leq \rho < p+1$ and $a \leq b + p - 1$ or
if $\rho = \sigma = p+1$, $q \leq p$ and $a \leq b + p - 1$,
2. at most $b + \rho + k$ if ρ does not satisfy 1 and if $a \leq b + \rho - 1$,
3. at most $a + k + 1$ if a does not satisfy 1 or 2.

(Always $p < \rho < p+1$ is true if ρ is not an integer.)

Case II. Let $f(z)$ have infinitely many positive, infinitely many negative and finitely many complex zeros ($f(z)$ may have a zero at $z=0$).

Let a_1 denote a real zero of $f(z)$ with multiplicity α_1 such that all real zeros of $B(z)$ are less than a_1 . All real zeros of $f(z)$ which are bigger than a_1 are denoted by a_n in increasing order and with multiplicities α_n ($n=2, 3, \dots$). Similarly let $-b_1$ denote a real zero of $f(z)$ with multiplicity β_1 such that all real zeros of $B(z)$ are bigger than $-b_1$. All real zeros of $f(z)$ which are smaller than $-b_1$ are denoted by $-b_n$ in decreasing order with multiplicities β_n ($n=2, 3, \dots$). If $B(z)$ does not have any real zeros, $-b_1$ and a_1 can be arbitrary real zeros of $f(z)$ with $-b_1 < a_1$.

Then at each point a_n and $-b_n$ ($n=1, 2, \dots$) $g(z)$ has a zero of exact multiplicity $\alpha_n - 1$ and $\beta_n - 1$ respectively and an odd number of zeros between each pair a_n, a_{n+1} and $-b_{n+1}, -b_n$ ($n=1, 2, \dots$).

Let k be the number of zeros of $f(z)$ which are different from a_n and $-b_n$ ($n=1, 2, \dots$).

Then the number of zeros of $g(z)$ besides the trivial zeros at a_n and $-b_n$ and besides one zero between each pair a_n, a_{n+1} and $-b_{n+1}, -b_n$ ($n=1, 2, \dots$) is

1. exactly $p+b+k$ if p is odd, $p \leq \rho < p+1$ and $a \leq b+p-1$, or
if $\rho = \sigma = p+1$, $p \geq q$ and $a \leq b+p-1$,
 2. exactly $p+b+k+1$ if p is even, $p \leq \rho \leq p+1$, $a \leq b+p$,
 3. at most $b+k+p$ if ρ is odd and does not satisfy 1 or 2, $a \leq b+p-1$,
 4. at most $b+k+p+1$ if ρ is even and does not satisfy 1 or 2, $a \leq b+p$,
 5. at most $a+k+1$ if $a-b$ is even and a does not satisfy 1-4,
 6. at most $a+k+2$ if $a-b$ is odd and a does not satisfy 1-4.
- (Always $p < \rho < p+1$ is true if ρ is not an integer.)

REMARKS. A similar result as in Case I holds for functions $f(z)$ which have infinitely many negative and finitely many positive zeros. One only has to replace $g(z)$ by $g(-z)$.

The results in Cases I and II remain correct if $B(z)$ does have real zeros of even order between zeros a_n, a_{n+1} and $-b_{n+1}, -b_n$ as long as they are different from all a_n and $-b_n$.

Proof. In the following we assume that $a_1 = 1$ and $-b_1 = -1$. This can be assumed without loss of generality, since we can replace the variable z by $\alpha z + \beta$ where α, β are suitable real constants.

From the assumptions made about $f(z)$ it follows that according to Hadamard's factorization theorem in Case I $f(z)$ can be written as $f(z) = K(z)e^{Q(z)}\Pi(z)$, where

$$\Pi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{\alpha_n} \exp \left[\alpha_n \left(\frac{z}{a_n} + \dots + \frac{1}{p} \left(\frac{z}{a_n} \right)^p \right) \right]$$

and in Case II as $f(z) = K(z)e^{Q(z)}\Pi_1(z)\Pi_2(z)$, where

$$\Pi_1(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{\alpha_n} \exp \left[\alpha_n \left(\frac{z}{a_n} + \dots + \frac{1}{p} \left(\frac{z}{a_n} \right)^p \right) \right]$$

and

$$\Pi_2(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{b_n}\right)^{\beta_n} \exp \left[\beta_n \left(-\frac{z}{b_n} + \dots + \frac{(-1)^p}{p} \left(\frac{z}{b_n} \right)^p \right) \right].$$

In Case I $K(z)$ is a real polynomial of degree k , whose zeros coincide with all zeros of $f(z)$ which are different from a_n ($n=1, 2, \dots$). $Q(z)$ is a real polynomial of degree q with $q \leq [p]$. p is the genus of the sequence a_n .

In Case II $K(z)$ is a real polynomial of degree k whose zeros coincide with all zeros of $f(z)$ which are different from a_n and $-b_n$ ($n=1, 2, \dots$). $Q(z)$ is as in Case I and p is defined by

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p} + \frac{\beta_n}{b_n^p} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^{p+1}} + \frac{\beta_n}{b_n^{p+1}} < \infty.$$

Substituting the expressions for $f(z)$ in $g(z) = A(z)f(z) + B(z)f'(z)$ we obtain in Case I

$$g = e^{\circ} \Pi (AK + BK' + BKQ' + BK(\Pi'/\Pi))$$

and in Case II

$$g = e^{\circ} \Pi_1 \Pi_2 [AK + BK' + BKQ' + BK((\Pi'_1/\Pi_1) + (\Pi'_2/\Pi_2))].$$

Here

$$\frac{\Pi'}{\Pi} = \sum_{n=1}^{\infty} \left(\frac{\alpha_n}{z-a_n} + \alpha_n \left(\frac{1}{a_n} + \cdots + \frac{z^{p-1}}{a_n^p} \right) \right) \quad \text{or} \quad \frac{\Pi'}{\Pi} = z^p \sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p(z-a_n)}.$$

Therefore in Case I

$$(2) \quad g = e^{\circ} \Pi \left(AK + BK' + BKQ' + BKz^p \left(\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p(z-a_n)} \right) \right)$$

and similarly in Case II

$$(3) \quad g = e^{\circ} \Pi_1 \Pi_2 \left(AK + BK' + BKQ' + BKz^p \left(\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p(z-a_n)} + (-1)^p \sum_{n=1}^{\infty} \frac{\beta_n}{b_n^p(z+b_n)} \right) \right).$$

Since in Case I $B(z)$ and $K(z)$ do not have zeros for $z \geq a_1$ it follows that $g(z)$ has an odd number of zeros between each pair a_n, a_{n+1} ($n=1, 2, \dots$). Similarly in Case II $B(z)$ and $K(z)$ do not have zeros for $z \geq a_1$ and $z \leq -b_1$, so that $g(z)$ has an odd number of zeros between each pair a_n, a_{n+1} and $-b_{n+1}, -b_n$ ($n=1, 2, \dots$).

From now on we assume without loss of generality that in Case I $B(z)K(z) > 0$ for $z \geq a_1$ (otherwise replace $g(z)$ by $-g(z)$) and in Case II that (see the definition of p in Case II) $\sum_{n=1}^{\infty} \alpha_n/a_n^p = \infty$ and $\sum_{n=1}^{\infty} \alpha_n/a_n^{p+1} < \infty$, and that at the same time $B(z)K(z) > 0$ for $z \geq a_1$ (otherwise replace $g(z)$ by $g(-z)$ and/or $g(z)$ by $-g(z)$).

Now in Case I

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p(z-a_n)} = - \sum_{n=1}^{\infty} \frac{1}{1-z(1/a_n)} \left(\frac{\alpha_n}{a_n^{p+1}} \right) = - \int_0^1 \frac{1}{1-zt} d\gamma(t)$$

with $\gamma(t) = \sum_{1/a_n \leq t} \alpha_n/a_n^{p+1}$ for $0 \leq t \leq 1$, $\gamma(t)$ being nondecreasing for $0 \leq t \leq 1$. This gives

$$g = e^{\circ} \Pi \left(AK + BK' + BKQ' - BKz^p \int_0^1 \frac{1}{1-zt} d\gamma(t) \right).$$

We now use Lemma 2 to obtain

$$B(z)K(z) \int_0^1 \frac{1}{1-zt} d\gamma(t) = P_{b+k-1}(z) + z^{b+k} \int_0^1 \frac{t^{b+k} B(1/t) K(1/t)}{1-zt} d\gamma(t)$$

where $P_{b+k-1}(z)$ is a real polynomial of degree $b+k-1$ at most. With $\eta(t) = \int_0^t \tau^{b+k} B(1/\tau) K(1/\tau) d\gamma(\tau)$, $\eta(t)$ being nondecreasing on $0 \leq t \leq 1$, we finally obtain in Case I

$$(4) \quad g = e^{\circ} \Pi \left(AK + BK' + BKQ' - z^p P_{b+k-1}(z) - z^{p+b+k} \int_0^1 \frac{1}{1-zt} d\eta(t) \right).$$

In Case II we put

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p(z-a_n)} = -\int_0^1 \frac{1}{1-zt} d\gamma_1(t) \quad \text{with } \gamma_1(t) = \sum_{1/a_n \leq t} \frac{\alpha_n}{a_n^{p+1}} \text{ for } 0 \leq t \leq 1,$$

$$\sum_{n=1}^{\infty} \frac{\beta_n}{b_n^p(z+b_n)} = \int_0^1 \frac{1}{1+zt} d\gamma_2(t) \quad \text{with } \gamma_2(t) = \sum_{1/b_n \leq t} \frac{\beta_n}{b_n^{p+1}} \text{ for } 0 \leq t \leq 1,$$

$\gamma_1(t), \gamma_2(t)$ being nondecreasing for $0 \leq t \leq 1$.

Now

$$\int_0^1 \frac{1}{1+zt} d\gamma_2(t) = \int_{-1}^0 \frac{1}{1-zt} d(-\gamma_2(-t))$$

and from (3) follows

$$(5) \quad g = e^Q \Pi_1 \Pi_2 \left(AK + BK' + BKQ' - BKz^p \times \left(\int_0^1 \frac{1}{1-zt} d\gamma_1(t) + \int_{-1}^0 \frac{1}{1-zt} d(-1)^p \gamma_2(-t) \right) \right).$$

Here $(-1)^p \gamma_2(-t)$ is increasing for $-1 \leq t \leq 0$ if p is odd and decreasing if p is even. Define

$$\gamma(t) = \begin{cases} -\gamma_2(-t) & \text{for } -1 \leq t \leq 0 \\ \gamma_1(t) & \text{for } 0 \leq t \leq 1 \end{cases} \quad \text{if } p \text{ is odd}$$

and

$$\gamma(t) = \begin{cases} \gamma_2(-t) & \text{for } -1 \leq t \leq 0 \\ \gamma_1(t) & \text{for } 0 \leq t \leq 1 \end{cases} \quad \text{if } p \text{ is even.}$$

Then (5) gives

$$(6) \quad g = e^Q \Pi_1 \Pi_2 \left(AK + BK' + BKQ' - BKz^p \int_{-1}^1 \frac{1}{1-zt} d\gamma(t) \right).$$

Using Lemma 2 again we have

$$B(z)K(z) \int_{-1}^1 \frac{1}{1-zt} d\gamma(t) = P_{b+k-1}(z) + z^{b+k} \int_{-1}^1 \frac{t^{b+k} B(1/t) K(1/t)}{1-zt} d\gamma(t)$$

where $P_{b+k-1}(z)$ is a real polynomial of degree $b+k-1$ at most.

If p is even we use Lemma 3 in addition to obtain

$$\int_{-1}^1 \frac{t^{b+k} B(1/t) K(1/t)}{1-zt} d\gamma(t) = c + z \int_{-1}^1 \frac{t^{b+k+1} B(1/t) K(1/t)}{1-zt} d\gamma(t)$$

where c is a real constant.

Finally we put

$$\eta(t) = \int_{-1}^t \tau^{b+k} B(1/\tau) K(1/\tau) d\gamma(\tau) \quad \text{for } -1 \leq t \leq 1 \text{ if } p \text{ is odd,}$$

$$\eta(t) = \int_{-1}^t \tau^{b+k+1} B(1/\tau) K(1/\tau) d\gamma(\tau) \quad \text{for } -1 \leq t \leq 1 \text{ if } p \text{ is even.}$$

Then we obtain from (6)

$$g = e^Q \Pi_1 \Pi_2 \left(AK + BK' + BKQ' - z^p P_{b+k-1} - z^{p+b+k} \int_{-1}^1 \frac{1}{1-zt} d\eta(t) \right) \quad \text{if } p \text{ is odd,}$$

$$(7) \quad g = e^Q \Pi_1 \Pi_2 \left(AK + BK' + BKQ' - z^p P_{b+k-1} - cz^{p+b+k} - z^{p+b+k+1} \int_{-1}^1 \frac{1}{1-zt} d\eta(t) \right) \quad \text{if } p \text{ is even.}$$

From the assumptions made about $B(z)K(z)$ without loss of generality it follows that $\eta(t)$ is an increasing function for $-1 \leq t \leq 1$.

In Case I we obtain

$$g = e^Q \Pi \left(P(z) - z^l \int_0^1 \frac{1}{1-zt} d\eta(t) \right)$$

and in Case II

$$g = e^Q \Pi_1 \Pi_2 \left(P(z) - z^l \int_{-1}^1 \frac{1}{1-zt} d\eta(t) \right) \quad \text{if } p \text{ is odd}$$

and

$$g = e^Q \Pi_1 \Pi_2 \left(P(z) - z^{l+1} \int_{-1}^1 \frac{1}{1-zt} d\eta(t) \right) \quad \text{if } p \text{ is even.}$$

Here $l=p+b+k$ and in all three cases $P(z)$ is a real polynomial.

To these three expressions we apply Lemma 3, if necessary, and then Lemma 1 in order to obtain that in C_1 and in C_2 respectively $g(z)$ has at most as many zeros as stated in Theorem 1.

If now the same proof is carried out with a_2 instead of a_1 , so that k has to be replaced by $k+\alpha_1$, the upper bound of zeros of $g(z)$ in C_1^* and C_2^* respectively, which is given by Lemma 1, is increased by α_1 . But in the statement of Theorem 1 these additional α_1 zeros are already counted with the trivial (α_1-1) -fold zero of $g(z)$ at a_1 and with one zero of $g(z)$ between a_1 and a_2 . Here the $*$ in C_1^* and C_2^* refers to the fact that now we assume without loss of generality that $a_2=1$.

This argument can be repeated with all a_n ($n=3, 4, \dots$) and all $-b_n$ ($n=2, 3, \dots$), which shows that besides the trivial zeros $g(z)$ has at most as many zeros as stated in Theorem 1.

It remains to show that in some special cases listed in Theorem 1 the upper bound for the number of zeros of $g(z)$ actually is the exact number of zeros.

In Case I with $l=p+b+k$ we obtained in (4) that

$$g(z) = e^{Q(z)} \Pi(z) \left(P_{l-1}(z) - z^l \int_0^1 \frac{1}{1-zt} d\eta(t) \right)$$

if $p \leq \rho < p+1$ and $a \leq b+p-1$, or $\sigma = \rho = p+1$, $q \leq p$, and $a \leq b+p-1$. Here $P_{l-1}(z)$ is a real polynomial of degree $l-1$ at most.

We now consider

$$h(z, \tau) = \tau P_{l-1}(z) - z^l \int_0^1 \frac{1}{1-zt} d\eta(t) \quad \text{for } 0 \leq \tau \leq 1.$$

Then

$$h(z, \tau) = \tau P_{l-1}(z) + z^p P_{b+k-1}(z) + B(z)K(z)z^p \sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p(z-a_n)},$$

which shows that for each τ , $0 \leq \tau \leq 1$, $h(z, \tau)$ has a simple pole at a_n and an odd number of zeros between each pair a_n, a_{n+1} ($n=1, 2, \dots$).

We now want to determine a circle S around the origin such that $h(z, \tau) \neq 0$ for $z \in S$ and $0 \leq \tau \leq 1$. With $z = re^{i\theta}$ we have for $r \geq 1$ and all $0 \leq \tau \leq 1$

$$\begin{aligned} \left| \frac{h(z, \tau)}{z^{l-1}} \right| &\geq \left| \operatorname{Im} \frac{h(z, \tau)}{z^{l-1}} \right| \geq \left| \operatorname{Im} \int_0^1 \frac{z(1-\bar{z}t)}{|1-zt|^2} d\eta(t) \right| - \left| \operatorname{Im} \tau \frac{P_{l-1}(z)}{z^{l-1}} \right| \\ &\geq r |\sin \theta| \left(\int_0^1 \frac{d\eta(t)}{|1-zt|^2} - \frac{c_1}{r^2} \right) \geq r |\sin \theta| \left(\int_0^1 \frac{t^{b+k} B(1/t) K(1/t)}{(1+rt)^2} d\gamma(t) - \frac{c_1}{r^2} \right) \\ &\geq r |\sin \theta| \left(c_2 \int_0^1 \frac{d\gamma(t)}{(1+rt)^2} - \frac{c_1}{r^2} \right) \quad \text{with } c_1, c_2 > 0, \end{aligned}$$

using the fact that $t^{b+k} B(1/t) K(1/t) \geq c_2 > 0$ for $0 \leq t \leq 1$.

Here

$$\int_0^1 \frac{d\gamma(t)}{(1+rt)^2} \geq \int_{1/r}^1 \frac{d\gamma(t)}{(1+rt)^2} \geq \int_{1/r}^1 \frac{d\gamma(t)}{(2rt)^2} \geq \frac{1}{4r^2} \int_{1/r}^1 \frac{d\gamma(t)}{t}.$$

Therefore

$$|h(z, \tau)| \geq |z|^{l-2} |\sin \theta| \left(\frac{c_2}{4} \int_{1/r}^1 \frac{d\gamma(t)}{t} - c_1 \right).$$

From the definition of $\gamma(t)$ follows

$$(8) \quad \int_{1/r}^1 \frac{d\gamma(t)}{t} = \sum_{r \geq a_n} \frac{\alpha_n}{a_n^p}$$

and therefore $\lim_{r \rightarrow \infty} \int_{1/r}^1 d\gamma(t)/t = \infty$. This shows that for $0 \leq \tau \leq 1$

$$(9) \quad |h(z, \tau)| > 0 \quad \text{if } r \text{ is sufficiently large and } \sin \theta \neq 0.$$

If $z = -r$ then for all $r \geq 1$ and $0 \leq \tau \leq 1$

$$|h(-r, \tau)| \geq r^{l-1} \left(r \int_0^1 \frac{1}{1+rt} d\eta(t) - c_3 \right) \geq r^{l-1} \left(c_2 \int_0^1 \frac{r}{1+rt} d\gamma(t) - c_3 \right)$$

which is > 0 if r is sufficiently large according to (8)⁽³⁾. The constants c_2, c_3 are > 0 .

⁽³⁾ Since $r \int_0^1 (d\gamma(t)/(1+rt)) \geq \frac{1}{2} \int_{1/r}^1 (d\gamma(t)/t)$.

If $z=r$, we choose r_1 large enough and close to, but $< a_m$ for some $m \geq 1$, so that (9) holds for $0 < \theta < 2\pi$ and such that

$$|h(r, \tau)| \geq \left| r^l \int_0^1 \frac{d\eta(t)}{1-rt} \right| - |P_{l-1}(r)| > 0 \quad \text{for } r_1 \leq r < a_m \text{ and } 0 \leq \tau \leq 1.$$

This is possible since $h(z, \tau)$ has a simple pole at each point a_n ($n=1, 2, \dots$). Let now S be the circle around the origin with this radius r_1 . Then $|h(z, \tau)| > 0$ for all $z \in S$ and $0 \leq \tau \leq 1$ so that $|h(z, \tau)| \geq c_4 > 0$ for these z and τ , since $|h(z, \tau)|$ is continuous as a function of $(z, \tau) \in S \times [0, 1]$.

Therefore $(1/2\pi i) \int_S (h'(z, \tau)/h(z, \tau)) dz$ is continuous as a function of τ for $0 \leq \tau \leq 1$ and therefore a constant. Since $h(z, 0)$ has an l -fold 0 at $z=0$ and at least one zero between each pair of poles a_n, a_{n+1} and since $h(z, \tau)$ has at least one zero between each pair a_n, a_{n+1} ($n=1, \dots, m-1$) for all $0 \leq \tau \leq 1$, we have

$$\frac{1}{2\pi i} \int_S \frac{h'(z, \tau)}{h(z, \tau)} dz \geq l.$$

Together with what has been proved earlier we obtain that $h(z, 1)$ or $g(z) = e^{Q(z)} \Pi(z) h(z, 1)$ has exactly l zeros besides one zero between each pair a_n, a_{n+1} ($n=1, 2, \dots$) and besides trivial zeros at a_n of multiplicity $\alpha_n - 1$.

In Case II with $l=p+b+k$ we have obtained in (7)

$$\begin{aligned} g(z) &= e^{Q(z)} \Pi_1 \Pi_2 \left(P_{l-1}(z) - z^l \int_{-1}^1 \frac{1}{1-zt} d\eta(t) \right) \quad \text{if } p \text{ is odd,} \\ g(z) &= e^{Q(z)} \Pi_1 \Pi_2 \left(P_l(z) - z^{l+1} \int_{-1}^1 \frac{1}{1-zt} d\eta(t) \right) \quad \text{if } p \text{ is even} \end{aligned}$$

and if the additional conditions of 1 or 2 in Case II of Theorem 1 are satisfied. Here $P_{l-1}(z)$ and $P_l(z)$ are real polynomials of degrees $l-1$ and l at most.

We now consider, similarly as in Case I for $0 \leq \tau \leq 1$

$$\begin{aligned} h(z, \tau) &= \tau P_{l-1}(z) - z^l \int_{-1}^1 \frac{1}{1-zt} d\eta(t) \\ (10) \quad &= \tau P_{l-1}(z) + z^p P_{b+k-1} + z^p B(z) K(z) \\ &\quad \times \left(\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p (z-a_n)} + (-1)^p \sum_{n=1}^{\infty} \frac{\beta_n}{b_n^p (z+b_n)} \right) \end{aligned}$$

if p is odd and

$$\begin{aligned} h(z, \tau) &= \tau P_l(z) - z^{l+1} \int_{-1}^1 \frac{1}{1-zt} d\eta(t) \\ (11) \quad &= \tau P_l(z) + cz^l + z^p P_{b+k-1} + z^p B(z) K(z) \\ &\quad \times \left(\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p (z-a_n)} + (-1)^p \sum_{n=1}^{\infty} \frac{\beta_n}{b_n^p (z+b_n)} \right) \end{aligned}$$

if p is even.

In both cases $h(z, \tau)$ has for each τ , $0 \leq \tau \leq 1$, a simple pole at a_n and $-b_n$ ($n=1, 2, \dots$) and an odd number of zeros between each pair a_n, a_{n+1} and $-b_{n+1}, -b_n$. Similarly as in Case I we now want to determine a curve S around the origin such that $h(z, \tau) \neq 0$ for $z \in S$ and $0 \leq \tau \leq 1$.

With $z = re^{i\theta}$ we have

$$(12) \quad \left| \frac{h(z, \tau)}{z^{l-1}} \right| \geq \left| \operatorname{Im} \frac{h(z, \tau)}{z^{l-1}} \right| \geq r |\sin \theta| \left(\int_{-1}^1 \frac{1}{|1-zt|^2} d\eta(t) - \frac{c_1}{r^2} \right) \quad \text{if } p \text{ is odd,}$$

$$(13) \quad \left| \frac{h(z, \tau)}{z^l} \right| \geq \left| \operatorname{Im} \frac{h(z, \tau)}{z^l} \right| \geq r |\sin \theta| \left(\int_{-1}^1 \frac{1}{|1-zt|^2} d\eta(t) - \frac{c_2}{r^2} \right) \quad \text{if } p \text{ is even.}$$

This holds for all $r \geq 1$ and all $0 \leq \tau \leq 1$ with constants c_1 and c_2 which are > 0 .

Now we use the fact that (w.l.o.g.) we have assumed $\sum_{n=1}^{\infty} \alpha_n/a_n^p = \infty$ and $\sum_{n=1}^{\infty} \alpha_n/a_n^{p+1} < \infty$ and that $t^{b+k}B(1/t)K(1/t) \geq c_3 > 0$ for $-1 \leq t \leq 1$. Then in both expressions (12), (13) we have if p is odd

$$\begin{aligned} \int_{-1}^1 \frac{1}{|1-zt|^2} d\eta(t) &= \int_{-1}^1 \frac{t^{b+k}B(1/t)K(1/t)}{|1-zt|^2} d\gamma(t) \geq c_3 \int_0^1 \frac{d\gamma_1(t)}{|1-zt|^2} \geq c_3 \int_0^1 \frac{d\gamma_1(t)}{(1+rt)^2} \\ &\geq c_3 \int_{1/r}^1 \frac{d\gamma_1(t)}{(2rt)^2} \geq \frac{c_3}{4r^2} \int_{1/r}^1 \frac{d\gamma_1(t)}{t}. \end{aligned}$$

Similarly if p is even

$$\int_{-1}^1 \frac{1}{|1-zt|^2} d\eta(t) = \int_{-1}^1 \frac{t^{b+k+1}B(1/t)K(1/t)}{|1-zt|^2} d\gamma(t) \geq c_3 \int_0^1 \frac{td\gamma_1(t)}{|1-zt|^2}$$

and now this is

$$\geq \frac{c_3}{4r^2} \int_{1/r}^1 \frac{td\gamma_1(t)}{t^2} = \frac{c_3}{4r^2} \int_{1/r}^1 \frac{d\gamma_1(t)}{t}.$$

Consequently we obtain for all $0 \leq \tau \leq 1$ and $r \geq 1$ from (12), (13)

$$|h(z, \tau)| \geq r^{l-2} |\sin \theta| \left(\frac{c_3}{4} \int_{1/r}^1 \frac{d\gamma_1(t)}{t} - c_1 \right) \quad \text{if } p \text{ is odd,}$$

$$|h(z, \tau)| \geq r^{l-1} |\sin \theta| \left(\frac{c_3}{4} \int_{1/r}^1 \frac{d\gamma_1(t)}{t} - c_2 \right) \quad \text{if } p \text{ is even.}$$

Similarly as in Case I $\int_{1/r}^1 d\gamma_1(t)/t = \sum_{a_n \leq r} \alpha_n/a_n^p$ and therefore $\lim_{r \rightarrow \infty} \int_{1/r}^1 d\gamma_1(t)/t = \infty$ so that

$$(14) \quad |h(z, \tau)| > 0 \quad \text{for all } 0 \leq \tau \leq 1 \text{ if } r \text{ is sufficiently large and } \sin \theta \neq 0.$$

We then choose r_1 large enough and close to, but $< a_{m_1}$ for some $m_1 \geq 1$, such that for all $0 \leq \tau \leq 1$, $|h(z, \tau)| > 0$ for $r_1 \leq z < a_{m_1}$ and such that (14) holds for all $r \geq r_1$.

Similarly we choose $r_2 \geq r_1$ such that $-r_2$ is close enough to but $> -b_{m_2}$ for some $m_2 \geq 1$ with $|h(z, \tau)| > 0$ for $-b_{m_2} < z \leq -r_2$ and all $0 \leq \tau \leq 1$.

Now we can define the curve S as follows:

$$S : z = r_1 e^{i\theta} \quad \text{for } -\pi/2 \leq \theta \leq \pi/2, \quad z = r_2 e^{i\theta} \quad \text{for } \pi/2 \leq \theta \leq 3\pi/2, \\ z = r e^{i(\pi/2)} \quad \text{for } r_1 \leq r \leq r_2, \quad z = r e^{-i(\pi/2)} \quad \text{for } r_1 \leq r \leq r_2.$$

Then $|h(z, \tau)| > 0$ for $z \in S$ and $0 \leq \tau \leq 1$ and since $|h(z, \tau)|$ is continuous as a function of $(z, \tau) \in S \times [0, 1]$ we actually have $|h(z, \tau)| \geq c_4 > 0$ so that

$$\frac{1}{2\pi i} \int_S \left(\frac{h'(z, \tau)}{h(z, \tau)} \right) dz$$

is constant with respect to τ for $0 \leq \tau \leq 1$. Similarly as in Case I this integral is $\geq l$ if p is odd and $\geq l+1$ if p is even. Together with what has been proved earlier it follows that $g(z)$ has exactly l zeros (exactly $l+1$ zeros) besides at least one zero between each pair a_n, a_{n+1} and $-b_{n+1}, -b_n$ and besides the trivial zeros at a_n and $-b_n$ of multiplicities $\alpha_n - 1$ and $\beta_n - 1$ respectively if p is odd (if p is even).

Finally we want to end this section by giving some examples which show that the results of Theorem 1, Case II cannot be sharpened.

Let $\varphi(z) = e^{(\beta/2)z^2} \cos z$ with real $\beta \neq 0$. Here $\rho = 2$ and $p = 1$. Furthermore let

$$\gamma(z) = \alpha \varphi(z) + z \varphi'(z) = e^{(\beta/2)z^2} ((\alpha + \beta z^2) \cos z - z \sin z) \quad \text{with real } \alpha \neq 0.$$

We now apply Theorem 1, Case II, 1 to $g(z) = z \sin z - (\alpha + \beta z^2) \cos z$, where in the notation of Theorem 1 $f(z) = \sin z$, $a = 1$, $b = 2$, $a_1 = \pi$, $-b_1 = -\pi$. From this theorem and from $g(-z) = g(z)$ follows that $g(z)$ has exactly one zero between each pair of consecutive positive and negative zeros of $\sin z$ and that $g(z)$ has exactly two complex zeros and exactly two real zeros in $(-\pi, \pi)$ if α, β have the same sign. In particular in $(-\pi/2, \pi/2)$ $g(z)$ has no zero if $\alpha, \beta < 0$ and exactly two zeros if $\alpha, \beta > 0$.

We now apply Theorem 1, Case II, 6 to $g(z) = -(\alpha + \beta z^2) \cos z + z \sin z$ where we consider $\cos z$ as $f(z)$, $a = 2$, $b = 1$, and $a_1 = \pi/2$, $-b_1 = -\pi/2$. Then $g(z)$ also has exactly one zero between each pair of consecutive positive and negative zeros of $\cos z$. Besides these zeros $g(z)$ has exactly four zeros (2 complex and 2 in $(-\pi/2, \pi/2)$) if $\alpha, \beta > 0$ and exactly two zeros (2 complex and no zeros in $(-\pi/2, \pi/2)$) if $\alpha, \beta < 0$.

Therefore if $\alpha, \beta > 0$ the upper bound, as given in Case II, 6 is reached by $g(z)$ and if $\alpha, \beta < 0$ it is not.

At the same time $\varphi(z)$ and $\gamma(z)$ are examples which show that Case II, 1 and Case II, 4 are best possible.

In Case II, 1 the condition $p \leq \rho < p+1$ cannot be replaced by $p \leq \rho \leq p+1$, since in the above example $\rho = 2$, $p = 1$, but if $\alpha, \beta > 0$, $\gamma(z)$ has *exactly* four zeros besides one between each pair of consecutive positive and negative zeros of $\varphi(z)$.

In addition this example shows that the upper bound, as given by Case II, 4 is reached. On the other hand if $\alpha, \beta < 0$ this upper bound is not reached.

Finally we want to show that Case II,3 and Case II,5 are best possible also. Let $\varphi(z) = e^{(\beta/3)z^3} \cos z$ with real $\beta \neq 0$. Here $\rho = 3$, $p = 1$. Furthermore let $\gamma(z) = A(z)\varphi(z) + B(z)\varphi'(z)$, where $A(z)$ and $B(z)$ are real polynomials of exact degrees 3 and 1 respectively.

Then $\gamma(z) = e^{(\beta/3)z^3} g(z)$ with $g(z) = (A(z) + \beta z^2 B(z)) \cos z - B(z) \sin z$ and we also assume that $P(z) = A(z) + \beta z^2 B(z)$ shall have degree 3 exactly. From Theorem 1, Case II,1, applied to $g(z)$ with $f(z) = \sin z$, $a = 1$, $b = 3$, $a_1 = \pi$, $-b_1 = -\pi$, follows that $g(z)$ has exactly five zeros besides one between each pair of consecutive positive and negative zeros of $\sin z$ if $P(z)$ has its real zeros in $(-\pi, \pi)$. If we choose $B(z)$ and $P(z)$ such that $B(-\pi/2) > 0$, $B(\pi/2) < 0$ and $P(0) > 0$, $P(\pi/4) - B(\pi/4) < 0$, $P(-\pi/4) + B(-\pi/4) < 0$, $P(-\pi) < 0$, $P(\pi) > 0$, then $g(z)$ has exactly four zeros in $(-\pi/2, \pi/2)$ and exactly five zeros in $(-\pi, \pi)$.

If we choose $B(z)$ and $P(z)$ such that $B(-\pi/2) < 0$, $B(\pi/2) > 0$ and $P(z) \equiv 0$ then $g(z)$ has exactly two zeros in $(-\pi/2, \pi/2)$.

Now Theorem 1, Case II,5 applied to $g(z)$ with $f(z) = \cos z$, $a = 3$, $b = 1$, $a_1 = \pi/2$, $-b_1 = -\pi/2$ shows that $g(z)$, for both choices of the pair $B(z)$, $P(z)$ also has exactly one zero between each pair of consecutive positive and negative zeros of $\cos z$. In addition the upper bound, as given by Case II,5 is reached by $g(z)$ if $g(z)$ has four zeros in $(-\pi/2, \pi/2)$.

At the same time $\varphi(z)$ and $\gamma(z)$ are examples which show that the upper bound, as given by Case II,3 is reached in one case, while it is not reached in the other case.

In a similar way one can find examples which show that Cases I,2 and 3 are best possible by using $\cos \sqrt{z}$ or $\sqrt{z} \sin \sqrt{z}$.

3. In the following let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ denote a *real*, nonconstant, entire function of finite order ρ and with $f(0) \neq 0$. Especially we want to restrict $f(z)$ to the following two types (for the definition of p and q see the beginning of §2):

$$(1) \quad f(z) = c \prod_{n=1}^{\infty} \left(1 - \frac{z}{c_n}\right) \quad \text{with} \quad \sum_{n=1}^{\infty} |c_n|^{-1} < \infty$$

(i.e. either $\rho < 1$ or $\rho = 1$ and $p = q = 0$),

$$(2) \quad f(z) = ce^{bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{c_n}\right) e^{z/c_n} \quad \text{with} \quad \sum_{n=1}^{\infty} |c_n|^{-2} < \infty$$

(i.e. either $\rho < 2$ or $\rho = 2$ and $q \leq p = 1$).

Here b and c shall be real and $c \neq 0$. Furthermore let $F(z) = \sum_{n=0}^{\infty} a_n G(n) z^n$, where $G(z)$ is a real entire function of the following two types:

(A) $G(z) = e^{\beta z} \prod_{n=1}^{\infty} (1 + z/\alpha_n) e^{-z/\alpha_n}$, where each $\alpha_n > 0$, β real and $\sum_{n=1}^{\infty} \alpha_n^{-2} < \infty$. It is also allowed that $G(z)$ has finitely many (negative) zeros only.

(B) $G(z) = (z - \alpha_1) \cdots (z - \alpha_m)$, where $\alpha_1, \dots, \alpha_m > 0$.

Then always $F(z)$ is a real entire function of order $\leq \rho$. Having some information on the zeros of $f(z)$ we want to gain information on the zeros of $F(z)$. The following theorems, with the exception of Theorem 4, can be deduced from Theorem 1. The proof of Theorem 4 is independent of Theorem 1.

THEOREM 2. *Let $f(z)$ have infinitely many zeros but only k zeros which are not > 0 .*

(a) *If $f(z)$ is of type (1) and $G(z)$ of type (A) then $F(z)$ has at most k zeros which are not > 0 .*

(b) *If $f(z)$ is of type (1) and $G(z)$ of type (B) then $F(z)$ has at most $k+m$ zeros which are not > 0 .*

If $f(z)$ is of type (2) and $G(z)$ of type (B) then $F(z)$ has at most $k+2m$ zeros which are not > 0 . Besides $F(z)$ is of the same type as $f(z)$ and has infinitely many positive zeros. A corresponding result can be obtained if $f(z)$ has k zeros which are not < 0 .

THEOREM 3. *Let $f(z)$ be of type (2) with infinitely many real zeros of both signs and $2k$ complex zeros.*

(a) *If $G(z)$ is of type (A) then $F(z)$ has at most $2k$ complex zeros.*

(b) *If $G(z)$ is of type (B) then $F(z)$ has at most $2k+2m$ complex zeros and is of the same type as $f(z)$. Besides $F(z)$ has infinitely many real zeros of both signs.*

THEOREM 4. *Let $f(z) = \exp(az^2)h(z)$ where $h(z)$ is of type (2) and $a \leq 0$. Furthermore let $f(z)$ have real zeros only (or no zeros at all). If $G(z)$ is of type (A) then $F(z)$ has real zeros only.*

THEOREM 5. *Let $f(z)$ have infinitely many zeros but only k zeros which are not > 0 . Furthermore assume that $G(z)$ is of type (B) having the positive zeros $\alpha_1, \dots, \alpha_m$. With $f_0(z) = f(z)$ and $f_j(z) = -\alpha_j f_{j-1}(z) + z f'_{j-1}(z)$ for $j = 1, \dots, m$ we assume that for each j , $f_j(z)$ has exactly one zero between each pair of consecutive positive zeros of $f_{j-1}(z)$ and no zero between $z=0$ and the smallest positive zero of $f_{j-1}(z)$.*

If $f(z)$ is of type (1) then $F(z)$ has exactly $k+m$ zeros which are not > 0 . If $f(z)$ is of type (2) and $p=1$ then $F(z)$ has exactly $k+2m$ zeros which are not > 0 .

In both cases $F(z)$ is of the same type as $f(z)$. A corresponding result holds if $f(z)$ has k zeros which are not < 0 .

THEOREM 6. *Let $f(z)$ be of type (2) with infinitely many real zeros of both signs and with exactly $2k$ complex zeros. Furthermore assume that $G(z)$ is of type (B) having the positive zeros $\alpha_1, \dots, \alpha_m$. With $f_0(z) = f(z)$ and $f_j(z) = -\alpha_j f_{j-1}(z) + z f'_{j-1}(z)$ for $j = 1, \dots, m$ we assume that for each j , $f_j(z)$ has exactly one zero between each pair of consecutive positive and negative zeros of $f_{j-1}(z)$ and that $f_j(z)$ has no zero between the smallest positive and the smallest negative zero of $f_{j-1}(z)$.*

Under these conditions $F(z)$ has exactly $2k+2m$ complex zeros. Besides $F(z)$ is of the same type as $f(z)$.

In order to prove Theorems 2–6 we need the following

LEMMA 4. *Let $g(z) = \alpha f(z) + z f'(z)$ with real $\alpha \neq 0$, a , b and $f(z)$ real.*

(1) If $f(z) = e^{bz}h(z)$ where $h(z)$ is of type (1), then $g(z) = e^{bz}h^*(z)$ where $h^*(z)$ is of type (1).

(2) If $f(z) = e^{az^2}h(z)$ where $h(z)$ is of type (2), then $g(z) = e^{az^2}h^*(z)$ where $h^*(z)$ is of type (2).

In both cases we assume that all but finitely many zeros of $h(z)$ are real. Then the same is true for $h^*(z)$. It is allowed that $h(z)$ has finitely many zeros only.

Proof. (1) We have $f'(z)/f(z) = b + \sum_{n=1}^{\infty} 1/(z - c_n)$ where all but finitely many c_n are real. As z tends to infinity along the imaginary axis $f'(z)/f(z)$ tends to b . According to Hadamard's factorization theorem $g(z)$ is of the same form as $f(z)$. It remains to show that $g'(z)/g(z)$ also tends to b as z tends to infinity along the imaginary axis. We have

$$(3) \quad \frac{g'}{g} = \frac{f'}{f} + \frac{(f'/f) + z(f'/f)'}{\alpha + z(f'/f)}.$$

Here $z(f'/f)'$ tends to 0 as z tends to infinity along the imaginary axis. In addition we have for $z = iy$, y real:

$$\left| \alpha + z \frac{f'}{f} \right| \geq |z| \left| b + \sum_{n=N+1}^{\infty} \frac{1}{z - c_n} \right| - \left| \sum_{n=1}^N \frac{z}{z - c_n} \right| - |\alpha|$$

where we assume that $c_1, \dots, c_N$ consist of all complex zeros of $f(z)$. Here the first term on the right side is

$$\geq |z| \left| \operatorname{Im} \sum_{n=N+1}^{\infty} \frac{1}{z - c_n} \right| = y^2 \sum_{n=N+1}^{\infty} \frac{1}{y^2 + c_n^2},$$

and this tends to infinity with y . Therefore g'/g also tends to b as y tends to infinity.

(2) Here we have

$$\frac{1}{z} \frac{f'}{f} = 2a + \frac{b}{z} + \frac{1}{z} \sum_{n=1}^{\infty} \left(\frac{1}{z - c_n} + \frac{1}{c_n} \right)$$

where again all but finitely many c_n are real. Therefore f'/zf tends to $2a$ as z tends to infinity along the imaginary axis. We divide equation (3) by z and observe that now $(f'/f)'$ tends to $2a$ as z tends to infinity along the imaginary axis and that for the denominator $\alpha + z(f'/f)$ we have for $a=0$ (the case $a \neq 0$ being trivial because of (3)):

$$(4) \quad \left| \alpha + z \frac{f'}{f} \right| \geq |z| \left| b + \sum_{n=1}^N \frac{1}{c_n} + \sum_{n=N+1}^{\infty} \left(\frac{1}{z - c_n} + \frac{1}{c_n} \right) \right| - \left| \sum_{n=1}^N \frac{z}{z - c_n} \right| - |\alpha|.$$

Here again we assume that $c_1, \dots, c_N$ consists of all complex zeros of $f(z)$. Now if $z = iy$ is purely imaginary, the first term on the right side of (4) is

$$\geq |z| \left| \operatorname{Im} \sum_{n=N+1}^{\infty} \frac{1}{z - c_n} \right|$$

which similarly as in the proof of (1) tends to infinity with y . This shows that g'/zg also tends to $2a$ as y tends to infinity.

Proof of Theorems 2 and 3. Let α be real and $\neq 0$. Observing that with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ one has $g(z) = \alpha f(z) + zf'(z) = \sum_{n=0}^{\infty} a_n(\alpha + n)z^n$ we first prove the statement of the Theorem for the function $g(z)$.

From $g(z) = \alpha f(z) + zf'(z)$ we see that always $g(z)$ has an odd (even) number of zeros between $z=0$ and the smallest positive and the smallest negative zero of $f(z)$ if $\alpha > 0$ ($\alpha < 0$).

To prove Theorem 2(a) we apply Theorem 1, Case I, 1 to $g(z)$ with $\alpha > 0$. Since $g(z)$ has at least one zero between $z=0$ and the smallest positive zero of $f(z)$ it follows that $g(z)$ has at most k zeros which are not > 0 .

In order to prove part (b) of Theorem 2 we apply Theorem 1, Case I, 1 to $g(z)$ again but now with $\alpha < 0$, so that $g(z)$ has at most $k+1$ ($k+2$) zeros which are not > 0 if $f(z)$ is of type (1) (type (2)).

In all cases $g(z)$ has infinitely many real zeros > 0 and according to Lemma 4 $g(z)$ is of the same type as $f(z)$. Therefore $g(z)$ satisfies the same conditions of Theorem 2 as $f(z)$ does, the only difference being that in part (b) k has to be replaced by $k+1$ ($k+2$) if $f(z)$ is of type (1) (type (2)).

In order to finish the proof of part (b) we apply the above arguments m times where successively α is replaced by $-\alpha_j$ ($j=1, \dots, m$). Then we obtain that

$$\sum_{n=0}^{\infty} a_n(n-\alpha_1) \cdots (n-\alpha_m)z^n \quad \text{or} \quad F(z) = \sum_{n=0}^{\infty} a_n G(n)z^n$$

has at most $k+m$ ($k+2m$) zeros which are not > 0 if $f(z)$ is of type (1) (type (2)).

To finish the proof of part (a) we repeat the above arguments with $\alpha_1, \dots, \alpha_l > 0$ and obtain that

$$\sum_{n=0}^{\infty} a_n(\alpha_1 + n) \cdots (\alpha_l + n)z^n$$

for each l has at most k zeros which are not > 0 . This remains true if we divide this series by $\alpha_1 \cdots \alpha_l$ and replace z by $z \exp(\beta - \sum_{j=1}^l \alpha_j^{-1})$ which gives the function $g_l(z) = \sum_{n=0}^{\infty} a_n G_l(n)z^n$ where $G_l(z)$ converges to $G(z)$ for each z as l tends to infinity. Since $|a_n G_l(n)|$ and $|a_n G(n)|$ are $\leq |a_n| e^{\beta n}$ it follows that $g_l(z)$ converges, uniformly on compact domains, to the entire function $F(z) = \sum_{n=0}^{\infty} a_n G(n)z^n$, which according to a theorem of Hurwitz [6] still has at most k zeros which are not > 0 .

Theorem 3 is proved in a similar way. The only difference is that now Theorem 1, Case II has to be applied instead of Case I.

Proof of Theorem 4. From the assumptions made about $f(z)$ it follows that

$$\frac{f'}{f} = 2az + b + \sum_{n=1}^{\infty} \left(\frac{1}{z - c_n} + \frac{1}{c_n} \right)$$

where all c_n are real. Then we have $g(z) = \alpha f(z) + zf'(z) = zf(z)\varphi(z)$ where

$$\varphi(z) = \frac{\alpha}{z} + \frac{f'}{f} = \frac{\alpha}{z} + 2a + b + \sum_{n=1}^{\infty} \left(\frac{1}{z - c_n} + \frac{1}{c_n} \right).$$

Suppose that z is not real, then

$$\frac{\varphi(z) - \varphi(\bar{z})}{z - \bar{z}} = -\frac{\alpha}{|z|^2} + 2a - \sum_{n=1}^{\infty} \frac{1}{|z - c_n|^2}$$

which is < 0 if $a \leq 0$ and $\alpha > 0$. Consequently $g(z)$ does not have complex zeros if $\alpha > 0$. According to Lemma 4 $g(z)$ is of the same type as $f(z)$ (with $a \leq 0$!) and in order to finish the proof one only has to repeat the arguments of the proof of Theorem 2.

Theorems 5 and 6 finally are proved in the same way as Theorem 2(b). In addition one has to use the fact that Theorem 1 gives the *exact* number of zeros which, under the assumptions made in Theorems 5 and 6, cannot be > 0 in Theorem 5 and which cannot be real in Theorem 6.

Next we want to apply the previous theorems in order to obtain Hurwitz's theorem on the zeros of Besselfunctions.

Let

$$J_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(\nu + n + 1)}$$

denote the Besselfunction of order ν , which is defined for all $z \neq 0$ and with $-\pi < \arg z < \pi$. We assume that ν is real and $\neq -1, -2, \dots$. The zeros of $J_{\nu}(z)$ are determined by the zeros of the entire function

$$\phi_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(\nu + n + 1)}.$$

This entire function is of order 1 as can be seen from the coefficients of the series. Therefore according to Hadamard's factorization theorem $p \leq 1$ (in fact $p = 1$, but this is not needed here). Furthermore $\phi_{\nu}(z)$ satisfies: $\phi_{\nu}(0) \neq 0$ and

- (i) $\phi'_{\nu}(z) = -(z/2)\phi_{\nu+1}(z)$,
- (ii) $2\phi_{\nu}(z) = 2(\nu+1)\phi_{\nu+1}(z) + z\phi'_{\nu+1}(z)$,
- (iii) $z\phi''_{\nu}(z) + (2\nu+1)\phi'_{\nu}(z) + z\phi_{\nu}(z) = 0$.

Since $\varphi_{\nu}(z) = \phi_{\nu}(2\sqrt{z})$ is an entire function of order $1/2$, again according to Hadamard's factorization theorem $\phi_{\nu}(z)$ has infinitely many zeros, which are simple because of (iii).

THEOREM 7 (HURWITZ'S THEOREM). (a) $\phi_{\nu}(z)$ has real zeros only if $\nu > -1$.

(b) If m is a natural number and $-(m+1) < \nu < -m$ then $\phi_{\nu}(z)$ has exactly $2m$ complex zeros.

(c) If m is odd $\phi_{\nu}(z)$ has exactly 2 purely imaginary zeros and if m is even $\phi_{\nu}(z)$ has no purely imaginary zeros.

Proof. (a) follows from Theorem 4 with $f(z) = e^{-(z/2)^2}$, ($\rho = 2$, $p = 0$) and $G(z) = 1/\Gamma(z/2 + \nu + 1)$ which for $\nu > -1$ is of type (A).

(b) follows from Theorem 6 with

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n + \nu + m + 1)}, \quad (\rho = 1, p \leq 1)$$

and $G(z) = (\nu + 1 + z/2) \cdots (\nu + m + z/2)$ which is of type (B). The remaining conditions made in Theorem 6 are satisfied here because of the properties (i) and (ii) with $\alpha_j = -2(\nu + j)$, $j = 1, \dots, m$.

(c) follows from $\text{sign } \phi_\nu(0) = \text{sign } (1/\Gamma(\nu + 1))$ and by using (i) and (ii) for purely imaginary z .

More generally we consider for arbitrary real α , $\alpha J_\nu(z) + zJ'_\nu(z) = (z/2)^\nu \psi_{\nu\alpha}(z)$ with $\psi_{\nu\alpha}(z) = (\alpha + \nu)\phi_\nu(z) + z\phi'_\nu(z)$. If, for the moment, we put $g(z) = z^\alpha J_\nu(z)$, then

$$g' = z^{\alpha-1}(\alpha J_\nu + zJ'_\nu),$$

and from the differential equation for $J_\nu(z)$ or $\phi_\nu(z)$ follows

$$(5) \quad (z^{1-2\alpha}g')' = -z^{1-2\alpha}(1 + (\alpha^2 - \nu^2)/z^2)g.$$

If $|\nu| \leq |\alpha|$ (5) shows that $g'(z)$ or $\psi_{\nu\alpha}(z)$ has exactly one zero between each pair of consecutive positive zeros of $g(z)$ or $\phi_\nu(z)$ and that $g'(z)$ has exactly one (has no) zero between $z=0$ and the smallest positive zero of $\phi_\nu(z)$ if $(\alpha + \nu) > 0$ (if $(\alpha + \nu) < 0$).

With these results Theorem 1, Case II, 1 gives the following

THEOREM 8. *Let $\phi_\nu(z)$ have exactly $2m$ complex zeros ($m \geq 0$). Then for any real α $\psi_{\nu\alpha}(z)$ has exactly $2m + 2$ zeros besides one zero between each pair of consecutive positive and negative zeros of $\phi_\nu(z)$.*

Especially $\psi_{\nu\alpha}(z)$ has at most $2m(2m + 2)$ complex zeros if $(\alpha + \nu) > 0$ (if $(\alpha + \nu) < 0$).

If furthermore $|\nu| \leq |\alpha|$, then $\psi_{\nu\alpha}(z)$ has exactly $2m$ (exactly $2m + 2$) complex zeros if $(\alpha + \nu) > 0$ (if $(\alpha + \nu) < 0$).

The following theorem again is an application of Theorem 1, Case II.

THEOREM 9. *Let $\Pi(z)$ be a canonical product of genus p with infinitely many real zeros of both signs, no complex zeros and no zero at $z=0$. Furthermore consider $g(z) = \alpha f(z) + z^l f'(z)$, where $f(z) = z^m \Pi(z)$ with $m \geq 0$, α real and ≥ 0 and l odd.*

(a) *If $\alpha > 0$ and $m \geq 0$ then $g(z)$ has exactly $p + l - 2$ ($p + l - 1$) complex zeros when p is odd (even).*

(b) *If $\alpha = 0$ and $m > 0$ then $g(z)$ has exactly $p - 1$ (p) complex zeros when p is odd (even).*

(c) *If $\alpha = m = 0$ then $g(z)$ has no complex zeros.*

REMARK. A similar result can be obtained if $\Pi(z)$ still has infinitely many real zeros but not infinitely many of both signs. One has to apply Theorem 1, Case I instead of Case II.

Proof. $g(z) = z^{l+p}f(z)\varphi_1(z)$ if p is even and $g(z) = z^{l+p-1}f(z)\varphi_2(z)$ if p is odd. Here

$$\varphi_1(z) = \frac{\alpha}{z^{l+p}} + \frac{m}{z^{p+1}} + \sum_{n=1}^{\infty} \frac{1}{a_n^p(z-a_n)} \quad \text{where } p \text{ is even,}$$

and

$$\varphi_2(z) = \frac{\alpha}{z^{l+p-1}} + \frac{m}{z^p} + \sum_{n=1}^{\infty} \left(\frac{1}{a_n^{p-1}(z-a_n)} + \frac{1}{a_n^p} \right) \quad \text{where } p \text{ is odd.}$$

Next we form the derivatives and see that $\varphi_1'(z)$ and $\varphi_2'(z)$ are <0 for real $z \neq 0$ and $\neq a_n$. Consequently $g(z)$ has exactly one zero between each pair of consecutive positive and negative zeros of $f(z)$. Furthermore if $\alpha > 0$ and $m \geq 0$ $g(z)$ has exactly one zero between $z=0$ and the smallest positive and negative zero of $f(z)$. In addition $g(z)$ has an m -fold zero at $z=0$ and thus according to Theorem 1, Case II we obtain (a).

Cases (b) and (c) are proved similarly.

COROLLARY. Let $\alpha > 0$ and l odd, then the functions $\alpha \sin z + z^l \cos z$ and $\alpha \cos z - z^l \sin z$ have exactly $l-1$ complex zeros.

REFERENCES

1. E. Borel, *Leçons sur les fonctions entières*, 2nd ed., Gauthier-Villars, Paris, 1921.
2. E. Hille, *Analytic function theory*, Vol. II, Blaisdell, Waltham, Mass., 1962.
3. N. Obreschkoff, *Verteilung und Berechnung der Nullstellen reeller Polynome*, Deutscher Verlag der Wissenschaften, Berlin, 1963.
4. A. Peyerimhoff, *On the zeros of power series*, Michigan Math. J. **13** (1966), 193–214.
5. G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, 2nd ed., Springer, Berlin, 1954.
6. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Univ. Press, London, 1939.

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